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Equipartition of Energy for Anisotropic Elastic Waves

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1. INTRODUCTION

In a previous paper [1] it has been shown that the kinetic and the strain energy of each component wave, for elastic waves in an anisotropic medium, become equal to one-half of the total energy as the time tends to infinity. This result has been achieved by generalizing the Riemann–Lebesgue Lemma of harmonic analysis to a modified Fourier-type integral.

In this paper we give a generalization of a Paley–Wiener type result, which ensures the above mentioned equipartition of energy in finite time. In fact we show that if the initial data have compact supports restricted to a sphere of radius R, then equipartition for each component wave occurs for $t > R/\underline{v}_n$, where $\underline{v}_n$ is the minimum phase velocity of the nth wave over all directions of propagation. In physical terms this means that the time at which equipartition for the nth wave occurs is the moment where the corresponding expanding interior lacuna first opens.

When the medium is isotropic the phase velocity is the same in all directions. The corresponding result for the common Fourier integral has been proved by Duffin [6] and used to prove equipartition of energy, for many physically interesting cases, by one of the authors [2, 3, 4].

In Section 2 we give the necessary information from [1] and in Section 3 we prove our main result which is split in a series of lemmas.

2. ANISOTROPIC ELASTIC WAVES

The elastic wave propagation in an anisotropic medium is governed by the equations

$$\ddot{u}_i - \frac{1}{\rho} c_{ijkl} u_{k,lj} = 0, \qquad i = 1, 2, 3, \tag{1}$$

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where $(u_1, u_2, u_3) = \mathbf{u}: \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$ is the displacement field, ρ is the constant mass density, the "dot" represents differentiation with respect to time, the indices that appear after the comma represent space differentiation with respect to the corresponding components and c_{ijkl} are the elasticities, which are the components of a fourth rank tensor and satisfy the symmetry relations

$$c_{ijkl} = c_{klij} = c_{jikl}. \tag{2}$$

The acoustic tensor

$$B_{ik} = c_{ijkl} a_j a_l, \tag{3}$$

where $\mathbf{a} = (a_1, a_2, a_3)$ is any vector, is positive definite. Summation over repeated indices is understood. We assume that the initial data

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \qquad \dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{u}_1(\mathbf{x}) \tag{4}$$

have continuous third order derivatives and compact supports. In particular we choose $R > 0$ so that

$$\operatorname{supp} \mathbf{u}_0 \subset B(\mathbf{0}; R) = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < R\}, \qquad \operatorname{supp} \mathbf{u}_1 \subset B(\mathbf{0}; R). \tag{5}$$

The strain energy is given by the integral

$$S(t) = \int_{\mathbb{R}^3} W(\mathbf{x}, t)\, d^3x, \tag{6}$$

where

$$W(\mathbf{x}, t) = \tfrac{1}{2} c_{ijkl} u_{i,j} u_{k,l} \tag{7}$$

is the strain-energy density function, while the kinetic energy is

$$K(t) = \tfrac{1}{2} \int_{\mathbb{R}^3} \rho \dot{u}_i \dot{u}_i d^3x. \tag{8}$$

The total energy is conserved and it is given by

$$E(t) = S(t) + K(t) = E(0), \qquad t \geqslant 0. \tag{9}$$

It is well known [5, 7] that for $t > 0$ there are three mutually orthogonal waves that propagate along each direction with different phase velocities.

In [1] it has been shown that the kinetic $K^n(t)$ and the strain energies $S^n(t)$ of each one of these waves are given by

$$K^n(t) = \tfrac{1}{4}\int_{\mathbb{R}^3} [|\mathbf{e}_n \cdot \hat{\mathbf{u}}_1|^2 + r^2 v_n^2 |\mathbf{e}_n \cdot \hat{\mathbf{u}}_0|^2]\, d^3\xi$$

$$+ \tfrac{1}{4}\int_{\mathbb{R}^3} [|\mathbf{e}_n \cdot \hat{\mathbf{u}}_1|^2 - r^2 v_n^2 |\mathbf{e}_n \cdot \hat{\mathbf{u}}_0|^2] \cos(2trv_n)\, d^3\xi$$

$$- \tfrac{1}{2}\int_{\mathbb{R}^3} \mathrm{Re}[rv_n(\mathbf{e}_n \cdot \hat{\mathbf{u}}_0)(\mathbf{e}_n \cdot \hat{\mathbf{u}}_1)^*] \sin(2trv_n)\, d^3\xi, \tag{10}$$

$$S^n(t) = \tfrac{1}{4}\int_{\mathbb{R}^3} [|\mathbf{e}_n \cdot \hat{\mathbf{u}}_1|^2 + r^2 v_n^2 |\mathbf{e}_n \cdot \hat{\mathbf{u}}_0|^2]\, d^3\xi$$

$$- \tfrac{1}{4}\int_{\mathbb{R}^3} [|\mathbf{e}_n \cdot \hat{\mathbf{u}}_1|^2 - r^2 v_n^2 |\mathbf{e}_n \cdot \hat{\mathbf{u}}_0|^2] \cos(2trv_n)\, d^3\xi$$

$$+ \tfrac{1}{2}\int_{\mathbb{R}^3} \mathrm{Re}[rv_n(\mathbf{e}_n \cdot \hat{\mathbf{u}}_0)(\mathbf{e}_n \cdot \hat{\mathbf{u}}_1)^*] \sin(2trv_n)\, d^3\xi, \tag{11}$$

respectively, for $n = 1, 2, 3$.

Where the functions $\hat{\mathbf{u}}_0(\boldsymbol{\xi})$ and $\hat{\mathbf{u}}_1(\boldsymbol{\xi})$ are the Fourier transforms of $\mathbf{u}_0(\mathbf{x})$ and $\mathbf{u}_1(\mathbf{x})$, $r = |\boldsymbol{\xi}|$, $\boldsymbol{\xi} = r\mathbf{a}$, $v_n(\mathbf{a})$ is the phase velocity of the nth wave associated with the direction $\mathbf{a}$ and $\mathbf{e}_n(\mathbf{a})$ is the corresponding unit polarization vector. The asterisk means complex conjugation. From (10) and (11) it is obvious that the total energy of each wave is conserved. It has also been proved [1] that the second and third integrals in (10), or (11), tend to zero, as $t \to +\infty$ which implies the asymptotic equipartition of the energy for each wave separately. When v_n is constant, the last two integrals in (10) become zero for $t > R/v_n$ [3]. In our case v_n is not constant and therefore the Fourier-type integrals in (10) and (11) are modified (nonharmonic) Fourier transforms for which Duffin's work [6] does not apply. In what follows we give the generalization of Duffin's result for our modified Fourier transform.

3. The Finite Time Equipartition

In what follows we assume that

$$\mathbf{u}_0 \in [C^3(\mathbb{R}^3)]^3, \qquad \mathbf{u}_1 \in [C^3(\mathbb{R}^3)]^3, \tag{12}$$

$$(\operatorname{supp} \mathbf{u}_0) \cup (\operatorname{supp} \mathbf{u}_1) \subset \bar{B}(0; R) = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leqslant R\}, \tag{13}$$

$$\hat{\mathbf{u}}_0(\boldsymbol{\xi}) = \hat{\mathbf{u}}_0(r\mathbf{a}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathbf{u}_0(\mathbf{x}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d^3x, \tag{14}$$

$$\hat{\mathbf{u}}_1(\boldsymbol{\xi}) = \hat{\mathbf{u}}_1(r\mathbf{a}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathbf{u}_1(\mathbf{x}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d^3x, \tag{15}$$

$$S_3 = \{\mathbf{a} \in \mathbb{R}^3 : |\mathbf{a}| = 1\}, \tag{16}$$

$$I_c^n(t) = \int_{\mathbb{R}^3} [|\mathbf{e}_n \cdot \hat{\mathbf{u}}_1|^2 - r^2 v_n^2 |\mathbf{e}_n \cdot \hat{\mathbf{u}}_0|^2] \cos(2trv_n)\, d^3\xi, \tag{17}$$

$$I_s^n(t) = \int_{\mathbb{R}^3} \mathrm{Re}[rv_n(\mathbf{e}_n \cdot \hat{\mathbf{u}}_0)(\mathbf{e}_n \cdot \hat{\mathbf{u}}_1)^*] \sin(2trv_n)\, d^3\xi, \tag{18}$$

where

$$v_n : S_3 \to \mathbb{R}, \qquad n = 1, 2, 3 \tag{19}$$

are continuous on S_3,

$$0 < \underline{v}_n = \min\{v_n(\mathbf{a}) : \mathbf{a} \in S_3\} < \max\{v_n(\mathbf{a}) : \mathbf{a} \in S_3\} = \bar{v}_n < +\infty \tag{20}$$

and

$$\mathbf{e}_n : S_3 \to S_3, \qquad n = 1, 2, 3 \tag{21}$$

are also continuous on S_3.

LEMMA 1. *If we extend r to the complex variable z in* (14), (15) *then the following estimates hold for every $z \in \mathbb{C}$*

$$|\hat{\mathbf{u}}_0(z\mathbf{a})| \leqslant \frac{\gamma e^{R|\mathrm{Im}\, z|}}{(1+|z|)^3}, \tag{22}$$

$$|\hat{\mathbf{u}}_1(z\mathbf{a})| \leqslant \frac{\delta e^{R|\mathrm{Im}\, z|}}{(1+|z|)^3}, \tag{23}$$

where γ and δ are constants.

Proof. In [8, Theorem 7.22(a)] the following inequality is proved

$$|\hat{\mathbf{u}}_0(z_1, z_2, z_3)| \leqslant \frac{\gamma e^{R|\mathrm{Im}(z_1, z_2, z_3)|}}{(1+|(z_1, z_2, z_3)|)^3} \tag{24}$$

for every $(z_1, z_2, z_3) \in \mathbb{C}^3$ and γ constant. In our case, where the extension to complex values takes place only for r and not for each component of ξ, formula (24) gives

$$\begin{aligned} |\hat{\mathbf{u}}_0(z\mathbf{a})| &= |\hat{\mathbf{u}}_0(za_1, za_2, za_3)| \\ &\leqslant \frac{\gamma e^{R|\mathrm{Im}(za_1, za_2, za_3)|}}{(1+|(za_1, za_2, za_3)|)^3} = \frac{\gamma e^{R|\mathrm{Im}\, z|}}{(1+|z|)^3}, \end{aligned} \tag{25}$$

since for $z = x + iy$

$$|\operatorname{Im}(za_1, za_2, za_3)| = [y^2(a_1^2 + a_2^2 + a_3^2)]^{1/2} = |\operatorname{Im} z|,$$
$$|(za_1, za_2, za_3)| = [(x^2 + y^2)(a_1^2 + a_2^2 + a_3^2)]^{1/2} = |z|. \tag{26}$$

Similarly we show (23). The proof of Lemma 1 is completed.

LEMMA 2. *The integrals $I_c^n(t)$ and $I_s^n(t)$ can be written as*

$$I_c^n(t) = \tfrac{1}{2}\int_{-\infty}^{+\infty}\int_{S_3} \mathscr{F}_e^n(r, \mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, d\mathbf{a}\, dr, \tag{27}$$

and

$$I_s^n(t) = \frac{1}{2i}\int_{-\infty}^{+\infty}\int_{S_3} \mathscr{F}_0^n(r, \mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, d\mathbf{a}\, dr, \tag{28}$$

where

$$\mathscr{F}_e^n(r, \mathbf{a}) = [|\mathbf{e}_n \cdot \hat{\mathbf{u}}_1|^2 - r^2 v_n^2 |\mathbf{e}_n \cdot \hat{\mathbf{u}}_0|^2]\, r^2 \tag{29}$$

is an even function of r, and

$$\mathscr{F}_0^n(r, \mathbf{a}) = \operatorname{Re}[rv_n(\mathbf{e}_n \cdot \hat{\mathbf{u}}_0)(\mathbf{e}_n \cdot \hat{\mathbf{u}}_1)^*]\, r^2 \tag{30}$$

is an odd function of r.

Proof. From (22) and (23) for $z = r \in \mathbb{R}$ we obtain

$$|\mathscr{F}_e^n(r, \mathbf{a})| \leqslant \left[\frac{\gamma}{(1+r)^6} + \frac{r^2 \bar{v}_n^2 \delta}{(1+r)^6}\right] r^2 \leqslant \frac{\gamma + \bar{v}_n^2 \delta}{(1+r)^2}, \tag{31}$$

and since

$$\int_0^{\infty}\int_{S_3} \frac{1}{(1+r)^2}\, d\mathbf{a} dr < +\infty, \tag{32}$$

we can apply Fubini's theorem to write (17) as

$$I_c^n(t) = \int_0^{\infty}\int_{S_3} \mathscr{F}_e^n(r, \mathbf{a}) \cos(2trv_n)\, d\mathbf{a} dr. \tag{33}$$

Similarly

$$|\mathscr{F}_0^n(r, \mathbf{a})| \leqslant r^3 \bar{v}_n \frac{\gamma\delta}{(1+r)^6} \leqslant \frac{\bar{v}_n \gamma\delta}{(1+r)^3}, \tag{34}$$

$$\int_0^{\infty}\int_{S_3} \frac{1}{(1+r)^3}\, d\mathbf{a} dr < +\infty, \tag{35}$$

and (18) can be written as

$$I_s^n(t) = \int_0^\infty \int_{S_3} \mathscr{F}_0^n(r, \mathbf{a}) \sin(2trv_n)\, d\mathbf{a}dr. \tag{36}$$

From (14) and (15) we obtain for $k = 0, 1$,

$$\hat{\mathbf{u}}_k(-r\mathbf{a}) = \hat{\mathbf{u}}_k(-\boldsymbol{\xi}) = \hat{\mathbf{u}}_k^*(\boldsymbol{\xi}) = \hat{\mathbf{u}}_k^*(r\mathbf{a}). \tag{37}$$

Also

$$\begin{aligned} |\mathbf{e}_n \cdot \hat{\mathbf{u}}_k(-r\mathbf{a})|^2 &= [\mathbf{e}_n \cdot \hat{\mathbf{u}}_k(-r\mathbf{a})][\mathbf{e}_n \cdot \hat{\mathbf{u}}_k^*(-r\mathbf{a})] \\ &= [\mathbf{e}_n \cdot \hat{\mathbf{u}}_k^*(r\mathbf{a})][\mathbf{e}_n \cdot \hat{\mathbf{u}}_k(r\mathbf{a})] = |\mathbf{e}_n \cdot \hat{\mathbf{u}}_k(r\mathbf{a})|^2. \end{aligned} \tag{38}$$

From (38) we obtain immediately that

$$\mathscr{F}_e^n(-r, \mathbf{a}) = \mathscr{F}_e^n(r, \mathbf{a}). \tag{39}$$

Furthermore

$$\begin{aligned} \mathscr{F}_0^n(-r, \mathbf{a}) &= \mathrm{Re}[-rv_n(\mathbf{e}_n \cdot \hat{\mathbf{u}}_0(-r\mathbf{a}))(\mathbf{e}_n \cdot \hat{\mathbf{u}}_1^*(-r\mathbf{a}))](-r)^2 \\ &= -\mathrm{Re}[rv_n(\mathbf{e}_n \cdot \hat{\mathbf{u}}_0^*(r\mathbf{a}))(\mathbf{e}_n \cdot \hat{\mathbf{u}}_1(r\mathbf{a}))]\, r^2 \\ &= -\mathscr{F}_0^n(r, \mathbf{a}), \end{aligned} \tag{40}$$

since two complex conjugate numbers have the same real part. Relations (39) and (40) give

$$\begin{aligned} I_c^n(t) &= \tfrac{1}{2} \int_0^{+\infty} \int_{S_3} \mathscr{F}_e^n(r, \mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, d\mathbf{a}dr \\ &\quad + \frac{1}{2} \int_0^{+\infty} \int_{S_3} \mathscr{F}_e^n(r, \mathbf{a})\, e^{-i2trv_n(\mathbf{a})}\, d\mathbf{a}dr \\ &= \tfrac{1}{2} \int_0^{+\infty} \int_{S_3} \mathscr{F}_e^n(r, \mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, d\mathbf{a}dr \\ &\quad - \tfrac{1}{2} \int_0^{-\infty} \int_{S_3} \mathscr{F}_e^n(r, \mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, d\mathbf{a}dr \\ &= \tfrac{1}{2} \int_{-\infty}^{+\infty} \int_{S_3} \mathscr{F}_e^n(r, \mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, d\mathbf{a}dr, \end{aligned} \tag{41}$$

and

$$\begin{aligned}
I_s^n(t) &= \frac{1}{2i}\int_0^{+\infty}\int_{S_3} \mathscr{F}_0^n(r,\mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, dadr \\
&\quad - \frac{1}{2i}\int_0^{+\infty}\int_{S_3} \mathscr{F}_0^n(r,\mathbf{a})\, e^{-i2trv_n(\mathbf{a})}\, dadr \\
&= \frac{1}{2i}\int_0^{+\infty}\int_{S_3} \mathscr{F}_0^n(r,\mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, dadr \\
&\quad - \frac{1}{2i}\int_0^{-\infty}\int_{S_3} \mathscr{F}_0^n(r,\mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, dadr \\
&= \frac{1}{2i}\int_{-\infty}^{+\infty}\int_{S_3} \mathscr{F}_0^n(r,\mathbf{a})\, e^{i2trv_n(\mathbf{a})}\, dadr.
\end{aligned} \tag{42}$$

This completes the proof of Lemma 2.

LEMMA 3. *Define the functions*

$$\begin{aligned}
\tilde{\mathscr{F}}_e^n(z,\mathbf{a}) = [&(\mathbf{e}_n\cdot\hat{\mathbf{u}}_1(z\mathbf{a}))(\mathbf{e}_n\cdot\hat{\mathbf{u}}_1^{\#}(z\mathbf{a})) \\
&- z^2 v_n^2(\mathbf{e}_n\cdot\hat{\mathbf{u}}_0(z\mathbf{a}))(\mathbf{e}_n\cdot\hat{\mathbf{u}}_0^{\#}(z\mathbf{a}))]\, z^2
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\tilde{\mathscr{F}}_0^n(z,\mathbf{a}) = \tfrac{1}{2} z^3 v_n [&(\mathbf{e}_n\cdot\hat{\mathbf{u}}_0(z\mathbf{a}))(\mathbf{e}_n\cdot\hat{\mathbf{u}}_1^{\#}(z\mathbf{a})) \\
&+ (\mathbf{e}_n\cdot\hat{\mathbf{u}}_0^{\#}(z\mathbf{a}))(\mathbf{e}_n\cdot\hat{\mathbf{u}}_1(z\mathbf{a}))],
\end{aligned} \tag{44}$$

where

$$\hat{\mathbf{u}}_k^{\#}(z\mathbf{a}) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} \mathbf{u}_k(\mathbf{x})\, e^{-iz\mathbf{a}\cdot\mathbf{x}}\, d^3x, \qquad k = 0, 1 \tag{45}$$

(note that $\hat{\mathbf{u}}_k^{\#} \neq \hat{\mathbf{u}}_k^{*}$). Then $\tilde{\mathscr{F}}_e^n(z,\mathbf{a})$ and $\tilde{\mathscr{F}}_0^n(z,\mathbf{a})$ are entire functions of z and their restriction to $\mathbb{R}$ coincides with $\mathscr{F}_e^n(r,\mathbf{a})$ and $\mathscr{F}_0^n(r,\mathbf{a})$, respectively.

Proof. Consider first the extensions

$$\hat{\mathbf{u}}_k(z\mathbf{a}) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} \mathbf{u}_k(\mathbf{x})\, e^{iz\mathbf{a}\cdot\mathbf{x}}\, d^3x, \qquad k = 0, 1. \tag{46}$$

Since $\mathbf{u}_k(\mathbf{x})$, $k = 0, 1$, have compact supports, the Lebesgue dominated convergence theorem implies that $\hat{\mathbf{u}}_k(z\mathbf{a})$, $k = 0, 1$, are continuous functions of $z \in \mathbb{C}$. If $\partial\Delta$ denotes the perimeter of a triangle in $\mathbb{C}$ then for the jth component of $\hat{\mathbf{u}}_k(z\mathbf{a})$, $k = 0, 1$, we obtain

$$\int_{\partial\Delta} \hat{u}_{jk}(z\mathbf{a})\, dz = (2\pi)^{-3/2} \int_{\partial\Delta}\int_{\mathbb{R}^3} u_{jk}(\mathbf{x})\, e^{iz\mathbf{a}\cdot\mathbf{x}}\, d^3x dz$$

$$= (2\pi)^{-3/2} \int_{\mathbb{R}^3} u_{jk}(\mathbf{x}) \int_{\partial\Delta} e^{iz\mathbf{a}\cdot\mathbf{x}}\, dz d^3x$$

$$= (2\pi)^{-3/2} \int_{\mathbb{R}^3} u_{jk}(\mathbf{x}) \cdot 0\, d^3x = 0, \tag{47}$$

where we have used the analyticity of $e^{iz\mathbf{a}\cdot\mathbf{x}}$ in $\mathbb{C}$ and Fubini's theorem to interchange the integrals, since $u_{jk}(\mathbf{x})\, e^{iz\mathbf{a}\cdot\mathbf{x}}$, $j = 1, 2, 3$, $k = 0, 1$, are continuous and have compact supports for $(\mathbf{x}, z) \in \mathbb{R}^3 \times \partial\Delta$. Therefore Morera's theorem implies the analyticity of $\hat{\mathbf{u}}_k(z\mathbf{a})$, $k = 0, 1$, in $\mathbb{C}$.

Similarly the functions $\hat{\mathbf{u}}_k^{\#}(z\mathbf{a})$, $k = 0, 1$ are entire, and so are the functions $\tilde{\mathscr{F}}_e^n(z, \mathbf{a})$ and $\tilde{\mathscr{F}}_0^n(z, \mathbf{a})$, since they are rational expressions of entire functions. By continuity we obtain finally that

$$\lim_{z\to r} \tilde{\mathscr{F}}_e^n(z, \mathbf{a}) = \mathscr{F}_e^n(r, \mathbf{a}), \qquad n = 1, 2, 3 \tag{48}$$

and

$$\lim_{z\to r} \tilde{\mathscr{F}}_0^n(z, \mathbf{a}) = \mathscr{F}_0^n(r, \mathbf{a}), \qquad n = 1, 2, 3, \tag{49}$$

which completes the proof of the lemma.

LEMMA 4. *For each $z \in \mathbb{C}$ the following estimates hold*

$$|\tilde{\mathscr{F}}_e^n(z, \mathbf{a})| \leqslant M_e^n \frac{e^{2R|\operatorname{Im} z|}}{(1 + |z|)^2}, \qquad n = 1, 2, 3, \tag{50}$$

and

$$|\tilde{\mathscr{F}}_0^n(z, \mathbf{a})| \leqslant M_0^n \frac{e^{2R|\operatorname{Im} z|}}{(1 + |z|)^3}, \qquad n = 1, 2, 3, \tag{51}$$

where M_e^n and M_0^n are constants.

Proof. From (22), (23), (29) and (30) we obtain

$$|\tilde{\mathscr{F}}_e^n(z, \mathbf{a})| \leqslant \left[\frac{\delta^2 e^{2R|\operatorname{Im} z|}}{(1 + |z|)^6} + \frac{|z|^2\, v_n^2\, \gamma^2\, e^{2R|\operatorname{Im} z|}}{(1 + |z|)^6} \right] |z|^2$$

$$\leqslant (\delta^2 + \bar{v}_n^2 \gamma^2) \frac{e^{2R|\operatorname{Im} z|}}{(1 + |z|)^2}, \tag{52}$$

which is (50) with $M_e^n = \delta^2 + \bar{v}_n^2 \gamma^2$. Also

$$|\mathscr{F}_0^n(z, \mathbf{a})| \leqslant \frac{|z|^3 \, v_n \gamma\delta \, e^{2R|\operatorname{Im} z|}}{(1+|z|)^6}$$
$$\leqslant \bar{v}_n \gamma\delta \frac{e^{2R|\operatorname{Im} z|}}{(1+|z|)^3}, \tag{53}$$

which is (51) with $M_0^n = \bar{v}_n \gamma\delta$. This completes the proof of the Lemma.

THEOREM. *For each $t > R/\underline{v}_n$ we have*

$$I_c^n(t) = I_s^n(t) = 0, \qquad n = 1, 2, 3. \tag{54}$$

Proof. We fix $\sigma > 0$ and apply Cauchy's theorem for the integrands $\mathscr{F}_e^n(z, \mathbf{a})\, e^{i2tzv_n(\mathbf{a})}$ and $\mathscr{F}_0^n(z, \mathbf{a})\, e^{i2tzv_n(a)}$ over the rectangle with vertices $r_0 + i0$, $r_0 + i\sigma$, $-r_0 + i\sigma$, $-r_0 + i0$. Since $\mathscr{F}_e^n(z, \mathbf{a})\, e^{i2tzv_n(\mathbf{a})}$ is entire, we obtain

$$\frac{1}{2}\left\{\int_{-r_0+i0}^{r_0+i0}\int_{S_3} + \int_{r_0+io}^{r_0+i\sigma}\int_{S_3} + \int_{r_0+i\sigma}^{-r_0+i\sigma}\int_{S_3} + \int_{-r_0+i\sigma}^{-r_0+i0}\int_{S_3}\right\}$$
$$\times \mathscr{F}_e^n(z, \mathbf{a})\, e^{i2tzv_n(\mathbf{a})}\, d\mathbf{a}\, dz = 0. \tag{55}$$

If we let $r_0 \to +\infty$ then the second and fourth integrals in (55) tend to zero since from (50) and $0 \leqslant \operatorname{Im} z \leqslant \sigma$ we obtain

$$|\mathscr{F}_e^n(z, \mathbf{a})\, e^{i2tzv_n(\mathbf{a})}|$$
$$\leqslant M_e^n \left(z^{2R|\operatorname{Im} z| - 2tv_n(\mathbf{a})\operatorname{Im} z} / (1 + \sqrt{r_0^2 + (\operatorname{Im} z)^2}\,)^2\right)$$
$$\leqslant M_e^n \frac{e^{2(R - t\underline{v}_n)\operatorname{Im} z}}{(1+|r_0|)^2} \xrightarrow[r_0 \to +\infty]{} 0. \tag{56}$$

Therefore for $n = 1, 2, 3$ we have

$$I_c^n(t) = \frac{1}{2}\int_{-\infty+i\sigma}^{+\infty+i\sigma}\int_{S_3} \mathscr{F}_e^n(z, \mathbf{a})\, e^{i2tzv_n(\mathbf{a})}\, d\mathbf{a}\, dz. \tag{57}$$

Similarly, since

$$|\mathscr{F}_0^n(z, \mathbf{a})\, e^{i2tzv_n(\mathbf{a})}| \leqslant M_0^n \frac{e^{2(R - t\underline{v}_n)\operatorname{Im} z}}{(1+|r_0|)^3} \xrightarrow[r_0 \to +\infty]{} 0, \tag{58}$$

we obtain

$$I_s^n(t) = \frac{1}{2i} \int_{-\infty + i\sigma}^{+\infty + i\sigma} \int_{S_3} \tilde{\mathscr{F}}_0^n(z, \mathbf{a})\, e^{i2tzv_n(\mathbf{a})}\, da dz \tag{59}$$

for $n = 1, 2, 3$. From (56) and (58) we obtain for $z = r + i\sigma$

$$|I_c^n(t)| \leqslant 4\pi M_e^n\, e^{2(R - t\underline{v}_n)\sigma}, \qquad n = 1, 2, 3 \tag{60}$$

and

$$|I_s^n(t)| \leqslant 2\pi M_0^n\, e^{2(R - t\underline{v}_n)\sigma}, \qquad n = 1, 2, 3. \tag{61}$$

If we let $\sigma \to +\infty$ in (60) and (61) we obtain (54) for each $t > R/\underline{v}_n$. This completes the proof of the theorem.

From the above theorem the last two integrals become zero for $t > R/\underline{v}_n$, which implies the equipartition

$$K^n(t) = S^n(t) = \tfrac{1}{2}E^n(0) \tag{62}$$

for each one of the component waves. Note that the equipartition is achieved at different times for each component wave. Equipartition of energy for the whole elastic wave is obtained for

$$t > \frac{R}{\min\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}}. \tag{63}$$

Remark. The above result holds true for any odd number of space dimensions.

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